Approximations to two real numbers

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Abstract. Probably we have observed a new simple phenomena dealing with approximations to two real numbers.

1. The result.

For a real ξ denote the irrationality measure function

$$\psi_\xi(t) = \min_{1\leqslant x\leqslant t} ||x\xi||.$$

Here we suppose x to be an integer number and $||\cdot||$ stands for the distance to the nearest integer.

The main result of this note is the following

Theorem 1. *For any two different irrational numbers α,β such that $\alpha\pm\beta\notin\mathbb{Z}$ the difference function*

$$\psi_\alpha(t)-\psi_\beta(t)$$

changes its sign infinitely many times as $t\to+\infty$.

The phenomenon observed in Theorem 1 cannot be generalized to any dimension greater than one. In [2] the following two statements were proven.

Theorem 2. (A. Khintchine, 1926) *Let function $\psi(t)$ decreses to zero as $t\to+\infty$. Then there exist two algebraically independent real numbers α^1,α^2 such that for all t large enough one has*

$$\psi_{(\alpha^1,\alpha^2)}(t) := \min_{1\leqslant \max(|x_1|,|x_2|)\leqslant t} ||x_1\alpha^1+x_2\alpha^2|| \leqslant \psi(t).$$

Theorem 3. (A. Khintchine, 1926) *Let $\psi_1(t)$ decreases to zero as $t\to+\infty$ and the function $t\mapsto t\psi_1(t)$ increases to infinity as $t\to+\infty$. Then there exist two algebraically independent real numbers α_1,α_2 such that for all t large enough one has*

$$\psi_{\left(\begin{array}{c}\alpha_1\\ \alpha_2\end{array}\right)}(t) := \min_{1\leqslant x\leqslant t}\max_{j=1,2}||x\alpha_j|| \leqslant \psi_1(t).$$

Of course in Theorems 2,3 we suppose x_1,x_2,x to be integers.

Take $\psi(t)=o(t^{-2})$, $t\to+\infty$. Take (β^1,β^2) to be numbers algebraically independent of α^1,α^2 such that they are badly approximable (in the sense of a linear form):

$$\inf_{(x_1,x_2)\in\mathbb{Z}^2\setminus\{(0,0)\}}\left(||x_1\beta^1+x_2\beta^2||\cdot\max(|x_1|,|x_2|)^2\right)>0.$$

We see that for all t large enough one has

$$\psi_{(\alpha^1,\alpha^2)}(t) < \psi_{(\beta^1,\beta^2)}(t).$$

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The similar situation holds in the case of simultaneous approximations. Take $\psi_1(t)=o(t^{-1/2}),\ t\to +\infty$. Take $\begin{pmatrix}\beta_1\\ \beta_2\end{pmatrix}$ to be numbers algebraically independent of α^1,α^2 such that they are badly simultaneously approximable:

$$\inf_{x\in\mathbb{Z}\setminus\{0\}}\left(\max_{j=1,2}||x\beta_j||\cdot|x|^{1/2}\right)>0.$$

We see that

$$\psi_{\begin{pmatrix}\alpha_1\\ \alpha_2\end{pmatrix}}(t)<\psi_{\begin{pmatrix}\beta_1\\ \beta_2\end{pmatrix}}(t)$$

for all t large enough. (Of course here $\psi_{(\beta^1,\beta^2)},\psi_{\begin{pmatrix}\beta_1\\ \beta_2\end{pmatrix}}$ are defined analogously to $\psi_{(\alpha^1,\alpha^2)},\psi_{\begin{pmatrix}\alpha_1\\ \alpha_2\end{pmatrix}}$.)

1. Proof of Theorem 1.

We can assume that $0<\alpha,\beta<1$. We consider continued fraction expansions

$$\alpha=[0;a_1a_2,,...,a_n,...],\ \ \beta=[0;b_1,b_2,...,b_n,...].$$

Define

$$\alpha_n=[a_n;a_{n+1},a_{n+2},...],\quad \alpha_n^*=[0;a_n,a_{n-1},...,a_1],$$
$$\beta_n=[b_n;b_{n+1},b_{n+2},...],\quad \beta_n^*=[0;b_n,b_{n-1},...,b_1],$$
$$\frac{r_n}{q_n}=[0;a_1,...,a_n],\quad \frac{s_n}{p_n}=[0;b_1,...,b_n].$$

Lemma 1. *For $n\geqslant 2$ one has*

$$||q_{n-1}\alpha||q_{n+1}=\frac{\alpha_{n+1}(a_{n+1}+\alpha_n^*)}{\alpha_{n+1}+\alpha_n^*}.$$

Proof.

It is a well known fact (see [1], Ch.1) that

$$\left|\alpha-\frac{r_{n-1}}{q_{n-1}}\right|=\frac{1}{q_{n-1}^2(\alpha_n+\alpha_{n-1}^*)},\qquad (1)$$

and

$$\alpha_n^*=\frac{q_{n-1}}{q_n}.$$

Instead of (1) we can write

$$||q_{n-1}\alpha||=\frac{1}{q_{n-1}\alpha_n+q_{n-2}}.\qquad (2)$$

So we see that

$$||q_{n-1}\alpha||q_{n+1}=||q_{n-1}\alpha||q_{n-1}\frac{q_n}{q_{n-1}}\frac{q_{n+1}}{q_n}=\frac{1}{(\alpha_n+\alpha_{n-1}^*)\alpha_n^*\alpha_{n+1}^*}.$$

But as

$$\alpha_n=a_n+\frac{1}{\alpha_{n+1}},\quad a_n+\alpha_{n-1}^*=\frac{1}{\alpha_n^*}$$

we see that

$$\alpha_n + \alpha^*_{n-1} = \frac{1}{\alpha^*_n} + \frac{1}{\alpha_{n+1}}.$$

So

$$||q_{n-1}\alpha||q_{n+1} = \frac{1}{\alpha^*_n\alpha^*_{n+1}\left(\frac{1}{\alpha^*_n} + \frac{1}{\alpha_{n+1}}\right)} = \frac{\alpha_{n+1}}{\alpha^*_{n+1}(\alpha^*_n + \alpha_{n+1})} = \frac{\alpha_{n+1}(a_{n+1} + \alpha^*_n)}{\alpha_{n+1} + \alpha^*_n}.$$

Lemma is proved.

As $a_{n+1} \geqslant 1$ and $\alpha_{n+1} > 1$ we obtain the following

Corollary. *For $n \geqslant 2$ one has*

$$||q_{n-1}\alpha||q_{n+1} > 1. \tag{3}$$

Lemma 2. *Suppose that $m, n \geqslant 2$ and*

$$q_{n+1} \leqslant p_{m+1}. \tag{4}$$

Then

$$||q_{n-1}\alpha|| > ||p_m\beta||. \tag{5}$$

Proof.

Suppose that (5) is not true. Then from (4) and (3) we see that

$$1 < ||q_{n-1}\alpha||q_{n+1} \leqslant ||p_m\beta||p_{m+1}.$$

As (see [1], Ch.1)

$$||p_m\beta||p_{m+1} = \frac{1}{1 + \frac{\beta^*_{m+1}}{\beta_{m+2}}} < 1$$

we have a contradiction. Lemma 2 is proved.

Now we are able to prove theorem 1.

Consider the sequences

$$q_0 \leqslant q_1 < \ldots < q_n < q_{n+1} < \ldots, \quad p_0 \leqslant p_1 < \ldots, < p_m < p_{m+1} < \ldots$$

of convergents' denominators to α, β correspondingly. Suppose that the statement of theorem 1 is false for certain irrationalities α, β. Without loss of generality assume that for all $t \geqslant p_{m_0} \geqslant q_{n_0-1}$ one has

$$\psi_\beta(t) \geqslant \psi_\alpha(t). \tag{6}$$

From Lemma 2 and the asumption (6) we see that between two consecutive denominators $p_m, p_{m+1}, m \geqslant m_0$ not more than one denominator of the form q_n may occur. Here we give a proof of this fact. Let $q_{n-1} \leqslant p_m < q_n < q_{n+1} < \ldots < q_{n+t} \leqslant p_{m+1}$ and $t \geqslant 1$. Then

$$||p_m\beta|| = \psi_\beta(p_m) \geqslant \psi_\alpha(p_m) = \psi_\alpha(q_{n-1}) = ||q_{n-1}\alpha||$$

and

$$q_{n+1} \leqslant q_{n+t} < p_{n+1}.$$

This contradicts to Lemma 2.

So we can define the sequence of integers

$$m_0 \geqslant 1, \;\; m_j \geqslant m_{j-1}+1$$

such that

$$p_{m_0} < q_{n_0} \leqslant p_{m_0+1} < ... < p_{m_1} < q_{n_0+1} \leqslant p_{m_1+1} < ... < p_{m_2} < q_{n_0+1} \leqslant p_{m_2+1} < ...$$

$$< p_{m_{j-1}} < q_{n_0+j-1} \leqslant p_{m_{j-1}+1} < ... < p_{m_j} < q_{n_0+j} \leqslant p_{m_j+1} < ... < p_{m_{j+1}} < q_{n_0+j+1} \leqslant p_{m_{j+1}+1} < ....$$

By (6) we see that for all $j \geqslant 0$ one has

$$||q_{n_0+j-1}\alpha|| = \psi_\alpha(q_{n_0+j-1}) = \psi_\alpha(p_{m_j}) \leqslant \psi_\beta(p_{m_j}) = ||p_{m_j}\beta||. \tag{7}$$

From (6) we also have

$$||q_{n_0+j}\alpha|| = \psi_\alpha(q_{n_0+j}) = \psi_\alpha(p_{m_j+1}) \leqslant \psi_\beta(p_{m_j+1}) = ||p_{m_j+1}\beta||. \tag{8}$$

We distinguish two cases. In the **first case** we suppose that for infinitely many j at least one of the inequalities in (7,8) is strict, that is there is the sign $<$ insead of $\leqslant$. In the **second case** for all j large enough we have equalities in both (7,8).

Consider the **first case**. Without loss of generality we assume that

$$||q_{n_0+j-1}\alpha|| = \psi_\alpha(q_{n_0+j-1}) = \psi_\alpha(p_{m_j}) < \psi_\beta(p_{m_j}) = ||p_{m_j}\beta||. \tag{9}$$

From (2) we have

$$||q_{n_0+j-1}\alpha|| = \frac{1}{q_{n_0+j-1}\alpha_{n_0+j} + q_{n_0+j-2}}, \quad ||p_{m_j}\beta|| = \frac{1}{p_{m_j}\beta_{m_j+1} + p_{m_j-1}}.$$

So

$$p_{m_j}\beta_{m_j+1} + p_{m_j-1} < q_{n_0+j-1}\alpha_{n_0+j} + q_{n_0+j-2} \tag{10}$$

As

$$\beta_{m_j+1} = b_{m_j+1} + \frac{1}{\beta_{m_j+2}}, \quad \alpha_{n_0+1} = a_{n_0+j} + \frac{1}{\alpha_{n_0+j+1}}$$

from (10) we deduce that

$$p_{m_j}\left(b_{m_j+1} + \frac{1}{\beta_{m_j+2}}\right) < q_{n_0+j-1}\left(a_{n_0+j} + \frac{1}{\alpha_{n_0+j+1}}\right) + q_{n_0+j-2}$$

or

$$p_{m_j+1} + \frac{p_{m_j}}{\beta_{m_j+2}} < q_{n_0+j} + \frac{q_{n_0+j-1}}{\alpha_{n_0+j+1}}.$$

But

$$p_{m_j+1} \geqslant q_{n_0+j}, \quad p_{m_j} \geqslant q_{n_0+j-1}.$$

So

$$\beta_{m_j+2} > \alpha_{n_0+j-1}. \tag{11}$$

From the other hand from (8) we deduce that

$$\frac{1}{q_{n_0+j}\alpha_{n_0+j+1} + q_{n_0+j-1}} = ||q_{n_0+j}\alpha|| = \psi_\alpha(q_{n_0+j}) = \psi_\alpha(p_{m_j+1}) \leqslant$$

$$\leqslant \psi_\beta(p_{m_j+1}) = ||p_{m_j+1}\beta|| = \frac{1}{p_{m_j+1}\beta_{m_j+2} + p_{m_j}}.$$

So

$$p_{m_j+1}\beta_{m_j+2} + p_{m_j} \leqslant q_{n_0+j}\alpha_{n_0+j+1} + q_{n_0+j-1}.$$

As

$$q_{n_0+j-1} \leqslant p_{m_j}, \quad q_{n_0+j} \leqslant p_{m_j+1}$$

we see that

$$\beta_{m_j+2} \leqslant \alpha_{n_0+j+1}.$$

This contradicts (11).

In the **second case** we see that for j large enough one has

$$\psi_\beta(p_{m_j+1}) = \psi_\alpha(q_{n_0+j}) = \psi_\beta(p_{m_{j+1}}).$$

Hence

$$m_j + 1 = m_{j+1}.$$

But in the case under consideration we see that there exist m_0, n_0 such that

$$p_{m_0+j}\beta - q_{n_0+j}\alpha = \pm r_{n_0+j} \pm s_{m_0+j},$$

$$p_{m_0+j+1}\beta - q_{n_0+j+1}\alpha = \mp r_{n_0+j+1} \mp s_{m_0+j+1},$$

where the choise of the signs $\pm$ depends on the lenghts of the corresponding continued fractons. Remind that α, β are irrational numbers. So

$$p_{m_0+j}q_{n_0+j+1} - p_{m_0+j+1}q_{n_0+j} = 0$$

and

$$\frac{p_{m_0+j}}{p_{m_0+j+1}} = [0; b_{m_0+j+1}, b_{m_0+j}, ..., b_1] = \frac{q_{n_0+j}}{q_{n_0+j+1}} = [0; a_{m_0+j+1}, a_{m_0+j}, ..., a_1], \quad j = 1, 2, 3, ...$$

and so $\alpha = \pm\beta$.

The proof of Theorem 1 is complete.

References

[1] W.M. Schmidt, Diophantine Approximations, Lect. Not. Math., 785 (1980).

[2] A.Y. Khinchine, Uber eine klasse linear Diophantine Approximationen // Rendiconti Circ. Math. Palermo, 50 (1926), p.170 - 195.